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SUMMARY

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I. Linear Programming Defined

One of the reasons why the programming tool has assumed importance, both in industry and in the military establishment, is that it is a method for studying the behavior of systems. In philosophy it is close to what some describe as the distinguishing feature of management science or operations research, to wit: "Operations are considered as an entity. The subject matter studied is not the equipment used, nor the morale of the participants, nor the physical properties of the output, it is the combination of these in total as an economic process."[*]

To many the term "linear programming" refers to mathematical methods for solving linear inequality systems. While this may be the central mathematical problem it is not its definition. Linear programming is a technique for building a model for describing the interrelations of the components of a system. As such it is probably the simplest mathematical model that can be constructed of any value for broad programming problems of industry and government. Thus the importance of the linear programming model is that it has wide applicability.

Suppose that the system under study (which may be one actually in existence or one which we wish to design) is a complex of machines, people, facilities, and supplies. It has certain

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[*]Operations Research for Management, C. C. Hermann and J. F. Magee, Harvard Bus. Rev., July, 1953.

over-all reasons for its existence. For the military it may be
to provide a striking force or for industry it may be to produce
certain types of products.

The linear programming approach is to consider the entire
system as decomposable into a number of elementary functions
called "activities"; each type of activity is abstracted to be
a kind of "black box" into which flow tangible things such as
supply and money and out of which may flow the products of manu-
facture or trained crews for the military. What goes on inside
the "box" is the concern of the engineer or the educator, but
to the programmer only the rates of flow in and out are of
interest.

The next step in building a model is to select some unit
for measuring the quantity of each activity. For a production
type activity it is natural to measure the quantity of the
activity by the amount of some product produced by it. This
quantity is called the activity level. To increase the activity
level it will be necessary, of course, to increase the flows into
and out of the activity. In the linear programming model the
quantities of flow of various items into and out of the activity
are always proportional to the activity level. Thus it is only
necessary to know the flows for the unit activity level. If we
wish to double the activity level, we simply double all the
corresponding flows for the unit activity level.

While any positive multiple of an activity is possible,
negative quantities of activities are not possible. The Mad
Hatter, you may recall in "Alice of Wonderland," was urging

Alice to have some more tea, and Alice was objecting that she couldn't see how she could take more when she hadn't had any. "You mean, you don't see how you can take less tea," said the Hatter, "it is very easy to take more than nothing." Lewis Carrol's point, of course, is that the activity of "taking tea" cannot be done in negative quantity.

One of the items in our system is regarded as precious in the sense that the total quantity of it produced by the system measures the payoff. The contribution of each activity to the total payoff is the amount of the precious item that flows into or out of each activity. Thus if the objective is to maximize profits, activities that require money contribute negatively and those that produce money contribute positively to total profits.

Next, it is required that the system of activities be complete in the sense that a complete accounting by activity can be made of each item. To be precise, for each item it is required that the total amount on hand equals the amount flowing into the various activities minus the amount flowing out. Thus, each item, in our abstract system, is characterized by a material balance equation — the various terms of which represent the flows into or out of the various activities.

The programming problem is to determine values for the levels which are positive or zero such that flows of each item (for these activity levels) satisfy the material balance equations and such that the value of the payoff is maximum. It is clear that what we have done is to reduce the programming problem to

a well-defined mathematical problem which is called the LINEAR
PROGRAMMING MODEL.

II. The L. P. Model Illustrated

To illustrate these principles of the linear programming
approach to model building, let us turn to an application in the
petroleum industry where linear programming methods have been
very successful. The complicated piece of plumbing of figure 1
is a simplified flow diagram of an oil refinery.[*] The problem
facing management is this. By turning valves, setting temper-
atures, pressures, and starting pumps, crude oil will be drawn
from one or several oil fields under the control of the refinery
(shown on the left). Like the old song about the music, it
"will go around and around" and come out as several streams of
pure oils (shown on the right). The latter can be marketed at
varying prices. By changing the controls, the quantities in
various streams of pure oils can be altered. This will change
the costs of operating the equipment and the revenues from the
sales of the final products. The various components are inter-
related, however, in such a complicated manner, that it is not
obvious what is the best way to operate the equipment to maximize
profits. In spite of these complex interrelations, when this
system is decomposed into elementary functions as the first step
in building a model, it turns out that there are essentially
only three main kinds of activities taking place: Distillation,
Cracking, Blending.

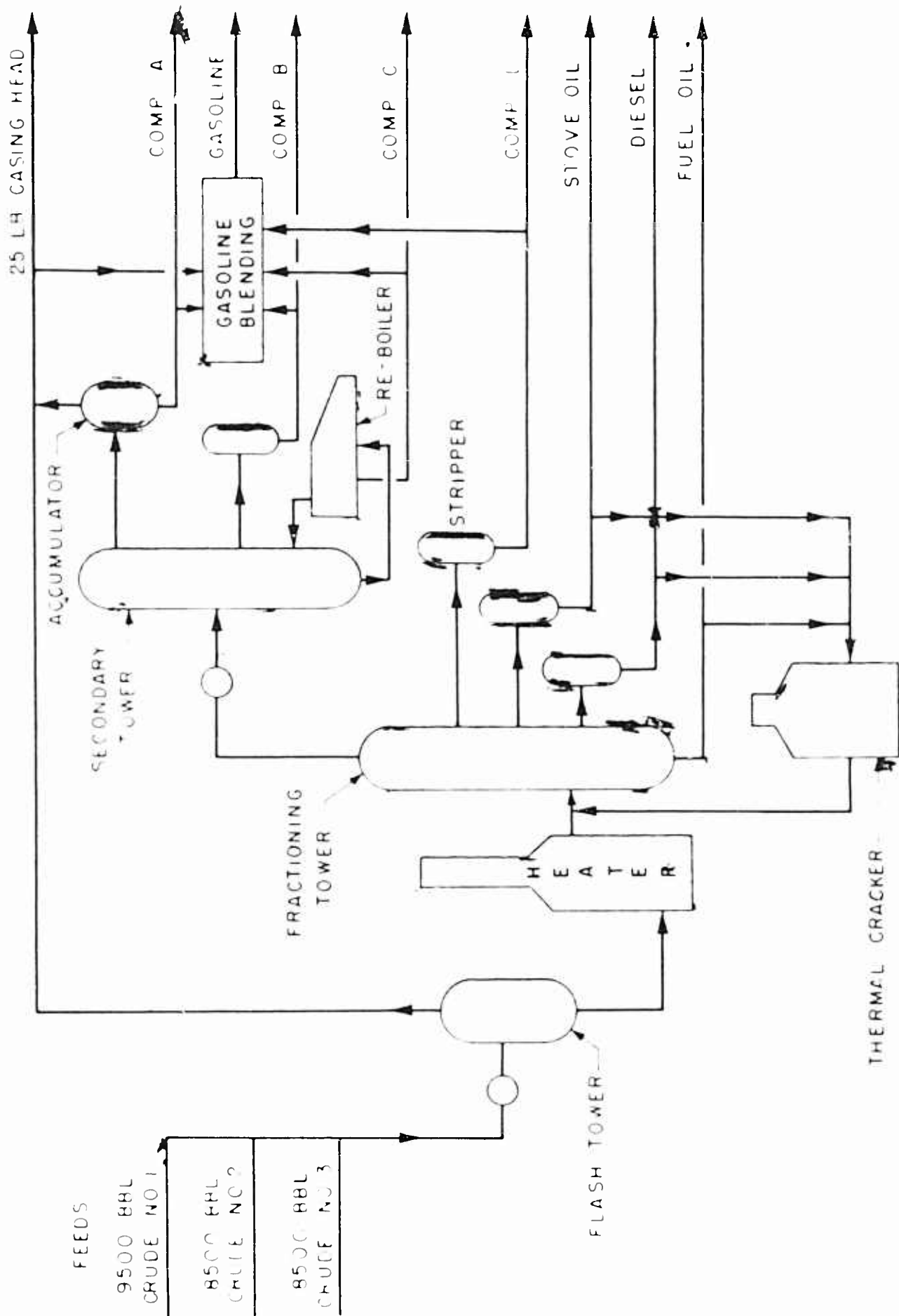[*]Refinery example taken from a term paper of R. J. Ullman.

FIG I - REFINERY FLOW DIAGRAM

Distillation Activity: The net effect of the flash tower,
heater, fractionating towers, strippers, etc., is to sep-
parate the crude into varying amounts of pure oils of which
it is composed. Crudes drawn from different oil fields
will have different decompositions. Hence there must be
separate distillation activity developed for each type
crude. The maximum amount of crude that can be distilled
depends on which, of the varying pieces of equipment it
passes through, will be the bottleneck. In our case we
will suppose it is the heater and that it has a fixed
capacity of 14,000 bbl's per day independent of type crude
processed. From this description it is evident if the level
of distillation activity is measured in number of barrels
of crude input, then a unit level of activity can be
pictures as in Figure 2a. It is seen that 1 bbl of Crude
No. 1 will use 1 bbl of distillation capacity, and will
cost $1.80 (to purchase and to distill); the outputs will
be a stream of pure oils in the amount shown. These out-
puts are princippally the heavier oils: fuel, diesel and
stove and smaller amounts of the lighter types used to
make gasoline. If instead of 1 bbl, it is desired to dis-
till 10 or X bbls of crude, all input and output quantities
of Figure 2a would have to be multiplied by 10 or X.

Cracking: The net effect of the cracking equipment is to take
one of the heavier type oils and to cause it to be broken
down into lighter type oils. In the case of fuel oil it will
produce a small amount of the lighter types and a larger
amount of stove oil which, if desired, can in turn be

recycled back into the cracker and made into lighter oils.
It is seen from Figure 2b that 1 unit of fuel oil requires
1 unit of cracking capacity, will cost $.16 and will produce
the pure oils in the amounts shown on the right. A separate
type activity must be set up for cracking of fuel, diesel
and stove oils.

Blending: Gasoline is not a pure oil but is a blend of several
of the lighter types of pure oil (see Figure 2c). It will
be noted the only output shown is the net revenue from
marketing 1 bbl of gasoline. The latter is assumed to be
the sales price at the refinery less the cost of the blending
operation.

Once the flows for these major activities have been deter-
mined on a per-barrel basis, it is a simple matter to set up the
linear programming model by means of which the managers can déter-
mine the best manner to operate the refinery to maximize profits.
In Figure 3 each column represents an activity. The input and
output quantities per unit level of activity are shown in the
column; to distinguish outputs from inputs, outputs are shown
with a minus sign. For example, the data of Figure 2a is shown
in column "Distillation — Crude 1"; the data of Figure 2b is
shown in column "Cracking — Fuel Oil"; the data of Figure 2c
is shown in column "Product Marketing — Gasoline." The other
activity columns are self-explanatory. The amounts available
of various items to the system are shown on the right.

The unknown activity levels to be determined are denoted by
x_1, x_2, ..., x_{20}. By multiplying these unknowns by the

DISTILLING
CRUDE
NO. I
(I BBL)

CRUDE NO I
I BBL

DIST CAPACITY
I BBL

COST
$I80

FUEL OIL .55
DIESEL OIL 20
STOVE OIL 10
COMP D .04
COMP C .04
COMP B .03
COMP A .02
CASINGHEAD .02

(a)

CRACKING
FUEL
OIL
(I BBL)

FUEL OIL
I BBL

CRACKING CAPACITY
I BBL

COST
$ 6

STOVE OIL 60
COMP D 06
COMP C 04
COMP B 04
COMP A 06
CASINGHEAD 12

(t)

BLENDING
GASOLINE
(I BBL)

COMP D 20
COMP C 30
COMP B 25
COMP A 10
CASINGHEAD 15

REVENUES
$ 5 50

(c)
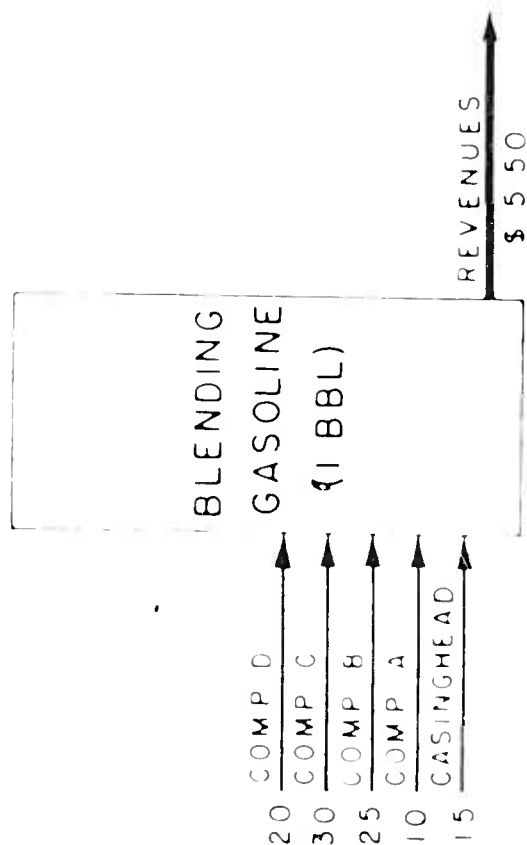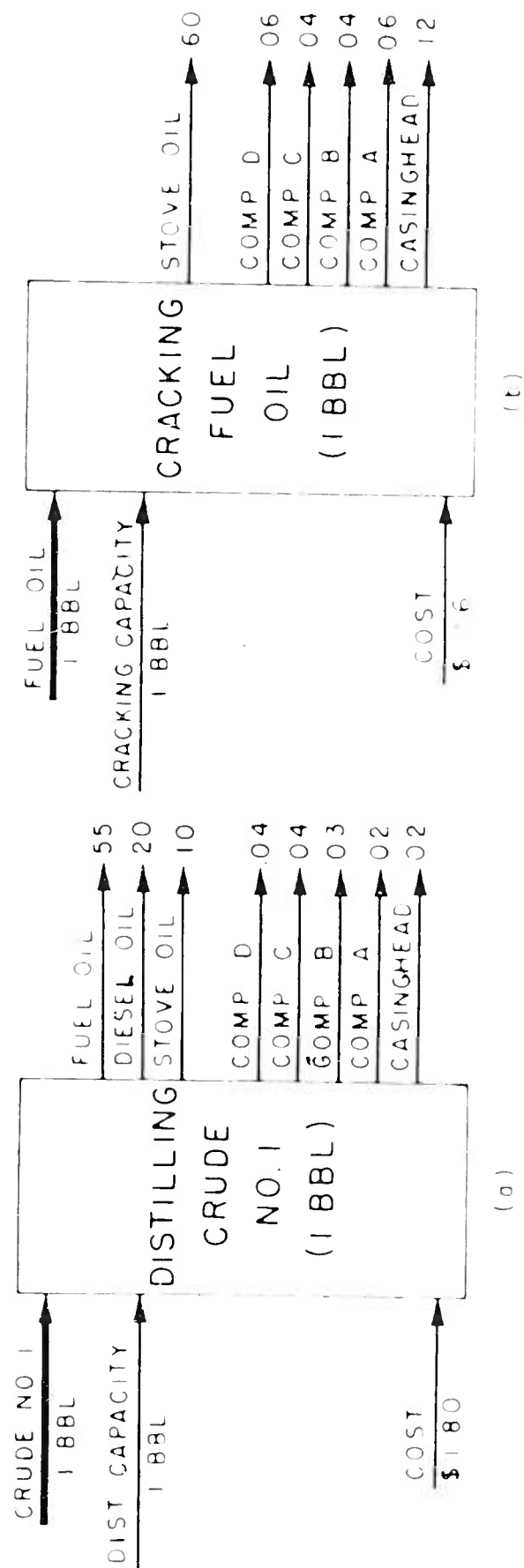
FIG 2

LINEAR PROGRAMMING MODEL OF A REFINERY

ITEMS	UNUSED			DISTILLATION				CRACKING				PRODUCT MARKETING									AVAILABLE (BBLS/DAY)
ACTIVITIES	Crude 1	Crude 2	Crude 3	Crude 1	Crude 2	Crude 3	Unused Crude Capacity	Fuel Oil	Diesel Oil	Stove Oil	Unused Crack. Capacity	Fuel Oil	Diesel Oil	Stove Oil	Gasoline	Comp. D	Comp. C	Comp. B	Comp. A	Casinghead	
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	
Crude 1	1			1																	= 9500
Crude 2		1			1																= 8500
Crude 3			1			1															= 8000
Crude Cap.				1	1	1	1														= 14000
Fuel Oil				-.05	-.61	-.50		1				1									= 0
Diesel Oil				-.20	-.12	-.11			1				1								= 0
Stove Oil				-.10	-.07	-.14		-.6	-.2	1				1							= 0
Crack Cap.								1	1	1	1										= 3500
Comp. D								-.06	-.41	-.30					.20	1					= 0
" C								-.04	-.20	-.30					.30		1				= 0
" B								-.04	-.04	-.04					.25			1			= 0
" A								-.06	-.12	-.10					.10				1		= 0
Casing-head								-.12	-.16	-.14					.15					1	= 0
Profit				-1.8	-1.9	-2.0		-.16	-.41	-.21		1.8	4.0	4.2	5.5	4.0	4.1	4.2	4.3	3.3	= MAXIMUM

FIG. 31

corresponding numbers found in any row and summing the terms across, the total obtained should equal the availability shown on the right.

For example, the first material balance equation reads

$$1.x_1 + 1.x_4 = 9500 ,$$

which means the amount of crude No. 1 available, 9500 bbls, is completely accounted for by the amount left in the ground, x_1, plus the amount distilled, x_4.

The fourth material balance equation, referring to the item distillation capacity, reads simply

$$1.x_4 + 1.x_5 + 1.x_6 + 1.x_7 = 14,000$$

which means that the distillation capacity of 14,000 barrels is completely accounted for by the amount used in distilling the various types of crude plus any excess capacity not used.

Finally the profit equation states the revenue obtained from marketing various products, $(1.8x_{12} + 4.0x_{13} + 4.2x_{14} + 5.5x_{15} + 4.0x_{16} + 4.1x_{17} + 4.2x_{18} + 4.3x_{19} + 3.3x_{20})$, less the cost of distilling and crude purchases, $(1.8x_4 + 1.9x_5 + 2.0x_6)$, less the cost of cracking, $(.16x_8 + .21x_9 + .21x_{10})$, is the amount of profit. The problem, of course, is to choose the program of activity levels in such a way that the material balance equations are satisfied and the profits maximized.